

# DANILOV RESOLUTION AND REPRESENTATIONS OF MCKAY QUIVER

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ABSTRACT. We attempt to generalize the classical McKay correspondence, concerning canonical Gorenstein singularities, to the terminal, non-Gorenstein case of  $\frac{1}{r}(1, a, r - a)$  singularity.

## 1. INTRODUCTION

For any finite subgroup  $G$  in  $\mathrm{SL}(n, \mathbb{C})$ , where  $n = 2, 3$ , the McKay correspondence established by [McK80, Rei02, Rei97, BKR01] connects the  $G$ -equivariant geometry of  $\mathbb{C}^n$  with the geometry of 'good' resolution of the quotient singularity  $\mathbb{C}^n/G$ . This resolution turns out to be crepant and isomorphic to the  $G$ -Hilbert scheme, that is a moduli space of scheme-theoretical  $G$ -orbits. The  $G$ -Hilbert scheme can be interpreted as a moduli space of semi-stable representations of the McKay quiver, which is an oriented graph coming from the representation theory of  $G$ .

The most remarkable result of Bridgeland, King and Reid says that the derived category of coherent sheaves on  $G$ -Hilbert scheme and the  $G$ -equivariant derived category of  $\mathbb{C}^n$  are equivalent. Moreover, for surfaces and 3-folds the  $G$ -Hilbert scheme turns out to be smooth and connected (see [BKR01]).

Let  $G = \langle \mathrm{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a}) \rangle$ , be a cyclic group of order  $r$  generated by a diagonal matrix, where  $\varepsilon = e^{\frac{2\pi i}{r}}$ , with  $a, r$  fixed, coprime natural numbers such that  $r > 1$ . The quotient singularity  $X = \mathbb{C}^3/G$  is the unique 3-dimensional cyclic, terminal quotient singularity (cf. [MS84]). We call this singularity of type  $\frac{1}{r}(1, a, r - a)$ .

In [Rei87] Reid defines the Danilov resolution, which is natural recursive resolution of  $\frac{1}{r}(1, a, r - a)$  singularity. It turns out that for  $a = \pm 1$  the Danilov resolution is isomorphic to a component of the  $G$ -Hilbert scheme containing free orbits (see [Kęd04]). However, for  $1 < a < r - 1$  the  $\mathrm{Hilb}^G \mathbb{C}^3$  scheme for the singularity of type  $\frac{1}{r}(1, a, r - a)$  is always singular (cf. [Kęd10]).

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The main idea behind this paper is to replace the singular  $G$ –Hilbert scheme by the Danilov resolution. In Theorem (9.2) we show that the Danilov resolution is isomorphic to the normalization of the coherent component of moduli space of representations of the McKay quiver for a suitable chosen stability parameter. Moreover, we give explicitly the cone of such stability conditions. We conjecture that the cone is a full chamber.

The paper is organized as follows. Section 3 recall definition of the Danilov resolution. In Section 4 we define effective divisors  $X_i, Y_i, Z_i$  which will be used in construction of a family of McKay quiver representations on the Danilov resolution. The McKay quiver is defined in Section 5. The family of quiver representations is constructed in Section 6 and we check that any two representations in that family are non-isomorphic. Section 7 recalls elementary facts on stability of quiver representations. In Section 8 we determine a cone of stability condition  $\theta$  for the constructed family. Finally, in Section 9 we prove the main theorem and compute explicitly the cone of stability conditions in the  $\frac{1}{5}(1, 2, 3)$  case.

## 2. NOTATION

Let  $G$  be a finite, cyclic subgroup of  $\mathrm{GL}(3, \mathbb{C})$  generated by a diagonal matrix  $\mathrm{diag}(\varepsilon, \varepsilon^a, \varepsilon^{r-a})$ , where  $\varepsilon = e^{\frac{2\pi i}{r}}$  and  $0 < a < r$  are coprime numbers. Let  $N_0 = \mathbb{Z}e_1 \oplus \mathbb{Z}e_2 \oplus \mathbb{Z}e_3$  be a free  $\mathbb{Z}$ –module with basis  $e_i$ . Denote by  $M_0 = \mathrm{Hom}_{\mathbb{Z}}(N_0, \mathbb{Z}) = \mathbb{Z}e_1^* \oplus \mathbb{Z}e_2^* \oplus \mathbb{Z}e_3^*$  the lattice dual to  $N_0$ , with  $e_i^*(e_j) = \delta_{ij}$ . Define  $N(r, a) = N_0 + \mathbb{Z}\frac{1}{r}(e_1 + ae_2 + (r-a)e_3)$  and  $M(r, a) = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ . The lattice  $M(r, a)$  can be identified with a sublattice of  $M_0$ , consisting of exponents of the  $G$ –invariant Laurent monomials. For any points  $p_1, \dots, p_n$  in the lattice  $N$  we denote by  $\langle p_1, \dots, p_n \rangle$  cone spanned by these points. For a rational polyhedral cone  $\sigma$  in  $N(r, a) \otimes \mathbb{R}$  by  $U_\sigma$  we mean the toric chart  $\mathrm{Spec}(\mathbb{C}[\sigma^\vee \cap M(r, a)])$ . By  $\bar{s}^t$  we mean the least non-negative integer  $u$  such that  $t$  divides  $s - u$ . Sometimes we just write  $\bar{s}$ , when  $t$  is obvious. All indices and all operations on vertices of McKay quiver are meant modulo  $r$ . By  $T$  we mean the torus  $\mathrm{Spec} \mathbb{C}[M]$ . The vector  $a_1e_1 + a_2e_2 + a_3e_3$  will be denoted  $(a_1, a_2, a_3)$ .

## 3. RECURSIVE DEFINITION OF DANILOV RESOLUTION

Let  $r$  and  $a$  be coprime, natural numbers, such that  $a < r$ . We recall definition of the Danilov resolution of the singularity  $\frac{1}{r}(1, a, r-a)$  (cf. [Rei87, p. 381]). Let  $\Delta(r, a) = \langle e_1, e_2, e_3 \rangle$  be the positive octant in  $N(r, a) \otimes \mathbb{R}$ . There exists ring isomorphism of  $\mathbb{C}[\Delta^\vee \cap M]$  with the ring of  $G$ –invariant regular functions on  $\mathbb{C}^3$ , therefore the quotient singularity  $X = \mathbb{C}^3/G$  is a toric variety given by the cone  $\Delta(r, a)$  in the lattice  $N(r, a)$ .

Let  $b$  denote the inverse of  $a$  modulo  $r$ . Set

$$p_i = \frac{1}{r}(\overline{-ib}^r, r-i, i) \quad \text{for } i = 0, \dots, r.$$

Note that  $p_{r-a} = \frac{1}{r}(1, a, r-a)$ . The following well-known lemma implies that the toric varieties associated to cones  $\langle e_1, e_2, p_{r-a} \rangle$ ,  $\langle e_1, e_3, p_{r-a} \rangle$  are isomorphic to the quotients of type  $\frac{1}{r-a}(1, \overline{r}^{r-a}, \overline{-r}^{r-a})$  and  $\frac{1}{a}(1, \overline{-r}^a, \overline{r}^a)$  respectively.

**Lemma 3.1.** *There exist  $\mathbb{Z}$ -linear isomorphisms*

$$L(r, a) : N(r-a, \overline{r}^{r-a}) \longrightarrow N(r, a),$$

$$R(r, a) : N(a, \overline{-r}^a) \longrightarrow N(r, a),$$

such that  $L(r, a)(\Delta(r-a, \overline{r}^{r-a})) = \langle e_1, e_2, p_{r-a} \rangle$ ,  $R(r, a)(\Delta(a, \overline{-r}^a)) = \langle e_1, e_3, p_{r-a} \rangle$ , and  $L(r, a)(e_1) = R(r, a)(e_1) = e_1$ ,  $L(r, a)(e_3) = R(r, a)(e_2) = p_{r-a}$ .

**Definition 3.2.** *The Danilov resolution of the singularity of type  $\frac{1}{r}(1, a, r-a)$  is a resolution obtained by a weighted blow-up of the toric cone  $\Delta(r, a)$  in the point  $p_{r-a}$  and, recursively, the Danilov resolutions of the singularities of type  $\frac{1}{r-a}(1, \overline{r}^{r-a}, \overline{-r}^{r-a})$  and  $\frac{1}{a}(1, \overline{-r}^a, \overline{r}^a)$ .*

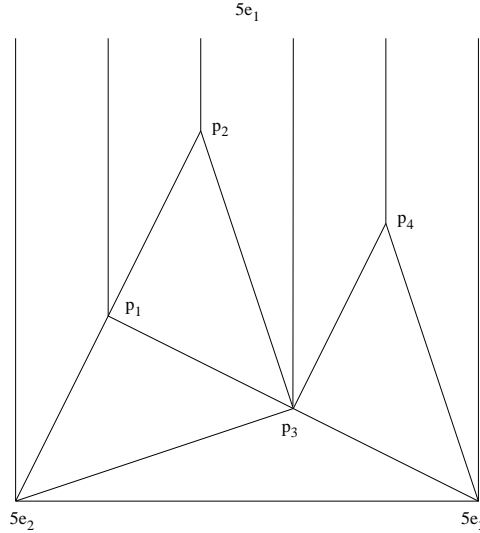


FIGURE 1. The fan of Danilov resolution of  $\frac{1}{5}(1, 2, 3)$  cut with hyperplane  $e_2^* + e_3^* = 5$ .

**Definition 3.3.** *For fixed  $r$  and  $a$ , we call the resolution of  $\frac{1}{r-a}(1, \overline{r}^{r-a}, \overline{-r}^{r-a})$  singularity an  $L$ -resolution and the resolution of  $\frac{1}{a}(1, \overline{-r}^a, \overline{r}^a)$  an  $R$ -resolution. The 3-dimensional cones of the fan of the Danilov resolution will be called  $L$ -cones or  $R$ -cones if they are subsets of the cones  $\langle e_1, e_2, p_{r-a} \rangle$  or  $\langle e_1, e_3, p_{r-a} \rangle$ , respectively.*

The fan of the Danilov resolution consist of  $2r-1$  simplicial, 3-dimensional cones. Precisely  $r$  cones of dimension 3 contain  $e_1$ . We call them  $\sigma_0, \dots, \sigma_{r-1}$ .

**Definition 3.4.** *Set*

$$\sigma_i = \langle p_i, p_{i+1}, e_1 \rangle, \quad \text{for } i = 0, \dots, r-1.$$

Note that the resolution can be constructed by  $r-1$  weighted blow-ups in the points  $p_1, \dots, p_{r-1}$  (up to the order).

**Corollary 3.5.** *The Danilov resolution of the singularity  $\frac{1}{r}(1, 1, r-1)$  is obtained by the consecutive blow-ups in the points  $p_{r-1}, \dots, p_1$ . In the case of  $\frac{1}{r}(1, r-1, 1)$  the blow-ups are made in the points  $p_1, \dots, p_{r-1}$ .*

**Definition 3.6.** *Let  $D_i$  denote the  $T$ -invariant toric divisor associated to the ray generated by the lattice point  $p_i$  for  $i = 0, \dots, r$ . Let  $E_j$  denote the  $T$ -invariant toric divisor associated to the ray generated by  $e_j$  for  $j = 1, 2, 3$ .*

Note that  $D_0 = E_2$  and  $D_r = E_3$ .

#### 4. DIVISORS $X_i, Y_i, Z_i$ AND THEIR PROPERTIES

In this section we start by defining recursively a permutation  $\tau$ , depending on  $a, r$ . It will be subsequently used in construction of divisors  $X_i, Y_i, Z_i$  on the Danilov resolution. These divisors will define the structure of a moduli space on the Danilov resolution.

**Definition 4.1.** *If  $a \in \{1, r-1\}$  set  $\tau(r, a, i) = \overline{ai-1}^r$  for  $i = 0, \dots, r-1$ , and otherwise*

$$\tau(r, a, i) = \begin{cases} \tau(r-a, \overline{r}^{r-a}, \overline{i}^{r-a}), & \text{if } i \geq a, \\ (r-a) + \tau(a, \overline{-r}^a, i), & \text{if } i < a. \end{cases}$$

Note that  $\tau(r, a, 0) = r-1$ .

The function  $\tau(r, a, \cdot)$  is a permutation of the set  $\{0, \dots, r-1\}$ . It will play a crucial role in determining the stability parameters connected with the moduli structure on the Danilov resolution.

**Definition 4.2.** *We call the sequence of numbers  $i, i+(r-a), \dots, i+s(r-a)$  an  $L$ -brick if*

- i)  $i < r-a$ ,
- ii)  $i+(s+1)(r-a) > r$ ,
- iii) every number in the sequence is strictly smaller than  $r$ .

*The sequence of numbers  $i, i+a, \dots, i+sa$  is called an  $R$ -brick if*

- 1)  $i < a$
- 2)  $i+(s+1)a > r$ ,
- 3) every number in the sequence is strictly smaller than  $r$ .

The  $L$ - and  $R$ -bricks connect the characters of a cyclic group of order  $r$  with the characters of cyclic groups of order  $r - a$  and  $a$ , respectively. They can be identified with fibers of the projections  $\mathbb{Z}/r\mathbb{Z} \longrightarrow \mathbb{Z}/a\mathbb{Z}$  and  $\mathbb{Z}/r\mathbb{Z} \longrightarrow \mathbb{Z}/(r - a)\mathbb{Z}$ .

There are  $a$  different  $R$ -bricks and  $r - a$  different  $L$ -bricks. For fixed  $r$  and  $a$  let  $Y$  denote the Danilov resolution of the  $\frac{1}{r}(1, a, r - a)$  singularity. The rest of this section is devoted to finding effective toric divisors  $X_i, Y_i, Z_i$  on  $Y$  for  $i = 0, \dots, r - 1$ . These divisors will be used directly in defining the structure of a moduli space on  $Y$ . Note that the addition in the indices of  $X_i, Y_i, Z_i$  is always meant modulo  $r$ .

**Definition 4.3.** *Let*

$$Y_{i-a} = \sum_{k=0}^{\tau(r,a,i)} D_k, \quad \text{for } i = 0, \dots, r - 1,$$

$$Z_i = \sum_{k=\tau(r,a,i)+1}^r D_k, \quad \text{for } i = 0, \dots, r - 1,$$

and let the divisor  $X_i$  be defined by the following equations:

$$X_i + Z_{i+1} = Z_i + X_{i-a}, \quad \text{for } i = 0, \dots, r - 1,$$

$$X_0 = E_1.$$

The last condition assures that the divisors  $X_i$  are uniquely determined. Note that by definition

$$Y_{i-a} + Z_i = \sum_{k=0}^r D_k,$$

that is  $Y_{i-a} + Z_i$  does not depend on  $i$ . Moreover

$$X_i + Y_{i+1} = Y_i + X_{i+a}.$$

The divisors  $X_i, Y_i, Z_i$  satisfy following commutativity relations:

$$(4.1) \quad X_i + Y_{i+1} = Y_i + X_{i+a},$$

$$(4.2) \quad X_i + Z_{i+1} = Z_i + X_{i-a},$$

$$(4.3) \quad Y_i + Z_{i+a} = Z_i + Y_{i-a}.$$

**Definition 4.4.** *For fixed  $a$  and  $r$ , by  $X_i^L, Y_i^L, Z_i^L$  we mean divisors on the  $L$ -resolution defined as in Definition (4.3) for  $r_L = r - a$  and  $a_L = \overline{r}^{r-a}$ . Similarly, by  $X_i^R, Y_i^R, Z_i^R$  we mean divisors on the  $R$ -resolution defined for  $r_R = a$  and  $a_R = \overline{-r}^a$ .*

Following propositions will prove useful in later sections.

**Lemma 4.5.** *Let  $i, \dots, i + s(r - a)$  be an  $L$ -brick. Restriction of the divisor  $Z_{i+s(r-a)}$  to the  $L$ -resolution is equal to the divisor  $Z_i^L$ . If  $i, \dots, i + sa$  is an  $R$ -brick then restriction of the divisor  $Z_i$  to the  $R$ -resolution is equal to the divisor  $Z_i^R$ .*

*Proof.* Observe that if  $i, \dots, i+s(r-a)$  is an  $L$ -brick, then  $i+s(r-a) \geq a$  and  $\overline{i+s(r-a)}^{r-a} = i$ . Therefore  $\tau(r, a, i+s(r-a)) = \tau(r-a, \overline{r}, i)$ . If  $i, \dots, i+sa$  is an  $R$ -brick then  $i < a$  and  $\tau(r, a, i) = (r-a) + \tau(a, \overline{-r}, i)$ .

Similar fact hold for restrictions of the divisors  $X_i$ .

**Lemma 4.6.** *For any  $i \leq r-2$  the divisor  $X_i$  restricted to the  $L$ -resolution is equal to the divisor  $X_j^L$ , where  $j = \overline{i}^{r-a}$ , and the divisor  $X_i$  restricted to the  $R$ -resolution is equal to the divisor  $X_j^R$ , where  $j = \overline{i}^a$ .*

*Proof.* We give the proof only in the case of restriction to the  $R$ -resolution. First we show that if  $i, \dots, i+sa$  is an  $R$ -brick such that  $i+sa \neq r-1$ , then the  $R$ -restrictions of the divisors  $X_i, \dots, X_{i+sa}$  are equal. To see this, observe that the restrictions of the divisors  $Z_j$  for  $j \geq a$  to the  $R$ -resolution are equal by definition of the permutation  $\tau$  and use the commutativity relation (4.2)

$$Z_j - Z_{j+1} = X_j - X_{j-a}$$

for  $j = i+a, i+2a, \dots, i+sa$ .

If  $i, \dots, i+sa$  is an  $R$ -brick such that  $i+sa = r-1$  then, by a similar proof, the restrictions of the divisors  $X_i, \dots, X_{i+(s-1)a}$  (i.e. all but the last) to the  $R$ -resolution are equal.

Hence, it is enough to prove the lemma assuming  $i < a$ . Denote by  $X_i|_R$  restriction of the divisor  $X_i$  to the  $R$ -resolution. Obviously  $X_0|_R = X_0^R$  and by Lemma (4.5) we obtain relations

$$X_i|_R + Z_{i+1}^R = Z_i^R + X_{i-a}|_R,$$

for  $i = 0, \dots, a-2$ . We have proven already that  $X_{i-a}|_R = X_{\overline{i+r}^a}|_R$  so the above relations can be rewritten as

$$X_i|_R + Z_{i+1}^R = Z_i^R + X_{\overline{i+r}^a}|_R.$$

These are exactly the equations (4.2) for  $r_R = a$  and  $a_R = \overline{-r}^a$ , so

$$X_i|_R = X_i^R \quad \text{for } i = 0, \dots, a-2.$$

Let  $j$  be the last element of an  $R$ -brick containing  $a-1$ . Then  $j \neq r-1$  so  $0 \leq \overline{j+a}^r < a-1$  and the equation (4.2)

$$X_{j+a} + Z_{j+a+1} = X_j + Z_{j+a},$$

restricted to the  $R$ -resolution becomes

$$X_{j+a-r}^R + Z_{j+a-r+1}^R = X_{a-1}|_R + Z_{j+a-r}^R.$$

This finishes the proof as the above is exactly the equation (4.2) for  $r_R = a, a_R = \overline{-r}^a$  and  $i = j+a-r$ .

By definition, the divisors  $Y_i - E_2$  and  $Z_i - E_3$  are effective for all  $i$ . Similar result holds for  $X_i - E_1$ .

**Lemma 4.7.** *The divisors  $X_i - E_1$  are effective for any coprime  $a, r$  and any  $i = 0, \dots, r-1$ .*

*Proof.* Note that for  $a \in \{1, r-1\}$  either  $X_i - E_1 = Y_i - E_2$  or  $X_i - E_1 = Z_i - E_3$ . For  $1 < a < r-1$ , by recursion and Lemma (4.6), the restrictions of  $X_i - E_1$  to  $L$ - and  $R$ -resolution are effective for  $i \neq r-1$ . Finally, note that

$$X_{r-1} - E_1 = (X_{r-a-1} - E_1) + (Z_{r-1} - Z_0),$$

where both summands are effective.

Define

$$\begin{aligned} D_X &= \sum_{i=0}^{r+1} e_1^*(p_i) D_i, \\ D_Y &= \sum_{i=0}^{r+1} e_2^*(p_i) D_i, \\ D_Z &= \sum_{i=0}^{r+1} e_3^*(p_i) D_i, \end{aligned}$$

where  $e_j^*(p_i)$  denotes the  $j$ -th coordinate of  $p_i$ .

We introduce the  $\mathbb{Q}$ -divisors  $R_i$  which play an auxiliary role. They will be not used until the proof of the Main Theorem.

**Definition 4.8.** For fixed  $r$  and  $a$ , define the  $\mathbb{Q}$ -divisors  $R_i$  for  $i = 0, \dots, r-1$  by the equations

$$\begin{aligned} Z_i &= D_Z + R_i - R_{i-a}, \\ R_0 &= 0. \end{aligned}$$

The divisors  $R_i$  are uniquely determined by the condition  $R_0 = 0$  since  $r, a$  are coprime and the rank of the matrix determining equations for  $R_i$  is equal to  $r-1$ . Using the equation  $Z_i + Y_{i-a} = D_Y + D_Z$  we get

$$Y_i = D_Y + R_i - R_{i+a}.$$

**Lemma 4.9.** For any coprime  $r$  and  $a$

$$R_1 = D_X - E_1.$$

*Proof.* By definition  $R_1 = (r-b)D_Z - (Z_0 + Z_{-a} + Z_{-2a} + \dots + Z_{r-b-1}) = (r-b)D_Z - (Z_{a+1} + Z_{2a+1} + Z_{3a+1} + \dots + Z_0)$ . Therefore it is enough to show that

$$E_1 = D_X - (r-b)D_Z - (Z_{a+1} + Z_{2a+1} + \dots + Z_0).$$

This holds by a recursive argument since the numbers in the sequence

$$(\star) \quad \overline{a+1}^r, \overline{2a+1}^r, \overline{3a+1}^r, \dots, \overline{0}^r,$$

not greater than  $a-1$  are equal to the numbers

$$\overline{a_R+1}^a, \overline{2a_R+1}^a, \overline{3a_R+1}^a, \dots, \overline{0}^a,$$

where  $a_R = \overline{-r}^a$ . Moreover, the numbers in the sequence  $(\star)$  not greater or equal to  $a$  are equal modulo  $r - a$  to the numbers

$$\overline{a+1}^{r-a}, \overline{2a_L+1}^{r-a}, \overline{3a_L+1}^{r-a}, \dots, \overline{0}^{r-a},$$

where  $a_L = \overline{r}^{r-a}$ . We omit a proof of this arithmetic fact. To finish note that the first coordinate of the point  $p_{i+1}$  is not smaller than the first coordinate of the point  $p_i$  if and only if the toric ray dual to the cone  $\langle p_i, p_{i+1} \rangle$  is equal to  $e_1^* - (r - b)e_3^*$ .

**Lemma 4.10.** *The  $\mathbb{Q}$ -divisors  $R_i$  satisfy*

$$X_i = D_X + R_i - R_{i+1} \quad \text{for } i = 0, \dots, r-1.$$

*Proof.* Set  $\widetilde{X}_i = D_X + R_i - R_{i+1}$  and note that  $\sum \widetilde{X}_i = rD_X$ . Moreover the divisors  $\widetilde{X}_i$  satisfy commutativity relations (4.2), hence  $X_i - \widetilde{X}_i$  is constant. Since  $\widetilde{X}_0 = D_X - R_1 = X_0$ , by Lemma (4.9), the constant is equal to 0.

**Corollary 4.11.** *The divisors  $R_i$  satisfy the following equations for  $i = 0, \dots, r-1$ .*

$$\begin{aligned} X_i &= D_X + R_i - R_{i+1}, \\ Y_i &= D_Y + R_i - R_{i+a}, \\ Z_i &= D_Z + R_i - R_{i-a}. \end{aligned}$$

## 5. THE MCKAY QUIVER

By a quiver we mean a finite, directed graph  $Q$ . The set of vertices of  $Q$  will be denoted by  $Q^0$  and the set of arrows by  $Q^1$ . For any arrow  $a$  in  $Q^1$  denote by  $tl(a)$  the tail of  $a$  and by  $hd(a)$  denote the head of  $a$ . The dimension vector  $\delta$  of quiver  $Q$  is a function  $\delta : Q^0 \rightarrow \mathbb{N}$ , assigning a natural number to every vertex of  $Q$ . Representation of the quiver  $Q$  with dimension vector  $\delta$  is an element of

$$\text{Rep}(Q, \delta) = \bigoplus_{a \in Q^1} \text{Hom}_{\mathbb{C}}(\mathbb{C}^{\delta(tl(a))}, \mathbb{C}^{\delta(hd(a))}),$$

hence it is a collection of  $\mathbb{C}$ -linear homomorphisms from  $\delta(tl(a))$ -dimensional vector space to  $\delta(hd(a))$ -dimensional vector space. By choosing a basis, we can identify those vector spaces with  $\mathbb{C}^{\delta(tl(a))}$  and  $\mathbb{C}^{\delta(hd(a))}$ . This allows to identify  $\text{Rep}(Q, \delta)$  with an affine scheme.

For any representation  $V \in \text{Rep}(Q, \delta)$  and  $a \in Q^1$  denote by  $V(a)$  a matrix representing arrow  $a$  in  $V$ .

A path  $q$  in quiver  $Q$  is a sequence of arrows  $a_l, \dots, a_2, a_1$  where  $hd(a_i) = tl(a_{i+1})$ . A linear combination of paths  $q_i$  is called an admissible relation, if paths  $q_i$  have the same heads and tails. Any set  $R$  of admissible relations for quiver  $Q$  defines an affine subscheme of  $\text{Rep}(Q, \delta)$  cut by the polynomial equations coming from  $R$  i.e.

$$\text{Rep}(Q, R, \delta) := \{V \in \text{Rep}(Q, \delta) \mid V(c) = 0 \text{ for } c \in R\},$$



where the function  $V(c)$  denotes the linear extension of the function  $V(q) = V(a_l) \cdot \dots \cdot V(a_1)$  defined for a path  $q$  consisting of arrows  $a_l, \dots, a_1$ .

If  $V, V'$  are representations of quiver  $Q$  with dimension vectors  $\delta, \delta'$  define a morphism  $f$  of representations  $V, V'$  as a set of  $\mathbb{C}$ -linear morphisms  $f(v) : \mathbb{C}^{\delta(v)} \rightarrow \mathbb{C}^{\delta'(v)}$ ,  $v \in Q^0$  satisfying the condition  $f(hd(a))V(a) = V'(a)f(tl(a))$  for every arrow  $a \in Q^1$ . We say that a sequence

$$0 \rightarrow V \xrightarrow{f} V' \xrightarrow{f'} V'' \rightarrow 0$$

is a short exact sequence of representations if the sequence

$$0 \rightarrow \mathbb{C}^{\delta(v)} \xrightarrow{f(v)} \mathbb{C}^{\delta'(v)} \xrightarrow{f'(v)} \mathbb{C}^{\delta''(v)} \rightarrow 0$$

is exact for every  $v \in Q^0$ .

Two representations of quiver  $Q$  are isomorphic if and only if they lie in the same orbit of the group

$$\mathrm{GL}(\delta, \mathbb{C}) = \bigoplus_{v \in Q^0} \mathrm{GL}(\delta(v), \mathbb{C}),$$

acting on the left on the set  $\mathrm{Rep}(Q, \delta)$  in the following way:

$$(g \cdot V)(a) = g(hd(a))V(a)g(tl(a))^{-1}, \quad \text{for any } V \in \mathrm{Rep}(Q, \delta).$$

This action leaves  $\mathrm{Rep}(Q, R, \delta)$  invariant. Dividing by the center we are left with a faithful action of the group

$$\mathrm{PGL}(\delta, \mathbb{C}) = \mathrm{GL}(\delta, \mathbb{C}) / \mathbb{C}^* \mathrm{Id},$$

where  $\mathrm{Id}$  stands for  $\#Q^0$ -tuple of identity matrices.

We do not need the general definition of the McKay quiver, so we quote only the specialization to the case of a cyclic group action.

**Definition 5.1.** (*McKay*) Let  $G$  be a cyclic group  $G \subset \mathrm{GL}(3, \mathbb{C})$  of order  $r$ , such that the quotient singularity  $\mathbb{C}^3/G$  is of type  $\frac{1}{r}(1, a, r-a)$ . Define McKay quiver for group  $G$  as a finite graph with  $r$  vertices  $0, 1, \dots, r-1$  and  $3r$  arrows  $x_0, y_0, z_0, \dots, x_{r-1}, y_{r-1}, z_{r-1}$  such that  $tl(x_i) = tl(y_i) = tl(z_i) = i$  and

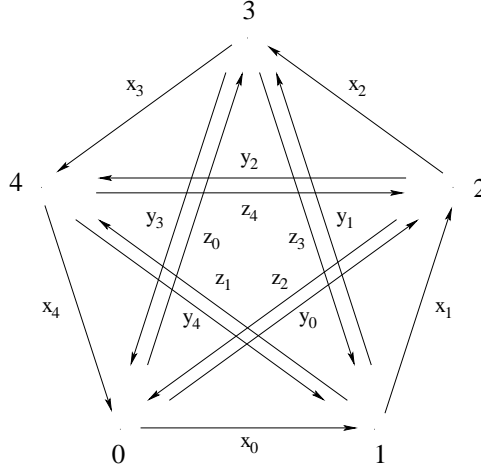
$$\begin{aligned} hd(x_i) &= \overline{i+1}^r, \\ hd(y_i) &= \overline{i+a}^r, \\ hd(z_i) &= \overline{i-a}^r. \end{aligned}$$

The vertices of McKay quiver correspond to the characters of  $G$ .

**Definition 5.2.** A representation of McKay quiver is an element of  $\mathrm{Rep}(Q, R, \delta)$ , where  $Q$  is the McKay quiver (for fixed  $r, a$ ) with constant dimension vector  $\delta \equiv 1$  and the relations

$$R = \{y_{i+1}x_i - x_{i+a}y_i, z_{i+1}x_i - x_{i-a}z_i, y_{i-a}z_i - z_{i+a}y_i \mid i = 0, \dots, r-1\},$$

where all indices are meant modulo  $r$ .

FIGURE 2. McKay quiver for  $r = 5, a = 2$ .

## 6. FAMILY OF REPRESENTATIONS OF MCKAY QUIVER

In this section we will define a family of McKay quiver representations over the Danilov resolution using line bundles determined by the effective divisors  $X_i, Y_i, Z_i$ .

**Definition 6.1** (King). *Fix coprime  $r, a$  and let  $X$  be any variety. By a family of McKay quiver representations on  $X$  for the action of type  $\frac{1}{r}(1, a, r - a)$  we mean a collection of  $3r$  line bundles  $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i$  on  $X$ , for  $i = 0, \dots, r - 1$  together with  $3r$  sections  $x_i, y_i, z_i$  satisfying commutativity relations*

$$x_i y_{i+1} = y_i x_{i+a},$$

$$x_i z_{i+1} = z_i x_{i-a},$$

$$y_i z_{i+a} = z_i y_{i-a},$$

for  $i = 0, \dots, r - 1$ .

**Definition 6.2.** *For fixed  $a$  and  $r$  set  $\mathcal{X}_i = \mathcal{O}(X_i)$ ,  $\mathcal{Y}_i = \mathcal{O}(Y_i)$ ,  $\mathcal{Z}_i = \mathcal{O}(Z_i)$ . The bundles  $\mathcal{X}_i, \mathcal{Y}_i, \mathcal{Z}_i$  together with natural sections define a family of McKay quiver representations on the Danilov resolution of  $\frac{1}{r}(1, a, r - a)$  singularity. We denote this family  $\mathcal{F}(r, a)$  (or  $\mathcal{F}$  in short).*

We will show that there exist stability conditions  $\theta$  such that every representation in the family  $\mathcal{F}(r, a)$  is  $\theta$ –(semi)stable. In fact, it will turn out that such stability conditions  $\theta$  are exactly those for which the representations parameterized by  $T$ –fixed point of the cones  $\sigma_0, \dots, \sigma_{r-1}$  are simultaneously  $\theta$ –(semi)stable.

**Definition 6.3.** *For any 3–dimensional cone  $\sigma$  in the fan of the Danilov resolution, we call an arrow in McKay quiver  $\sigma$ –distinguished if the section in the family  $\mathcal{F}$  corresponding to it does not vanish in the  $T$ –fixed point of toric chart  $U_\sigma$ .*

Observe, for example, that no  $z_i$ -arrow is  $\sigma_{r-1}$ -distinguished since the divisors  $Z_i - E_3$  are effective for any  $i$ .

**Lemma 6.4.** *For any 3-dimensional cone  $\sigma$  in the fan of the Danilov resolution any two vertices of the McKay quiver can be connected by an undirected path of  $\sigma$ -distinguished arrows different from  $x_{r-1}$ .*

*Proof.* The lemma is true for  $a \in \{1, r-1\}$ . Note that any two vertices of the McKay quiver lying in the same  $L$ -brick can be joined by a sequence of  $z$ -arrows for any  $L$ -cone  $\sigma$ , and any two vertices lying in the same  $R$ -brick can be joined by a sequence of  $y$ -arrows if  $\sigma$  is an  $R$ -cone. By the inductive step, any two bricks can be joined by a sequence of  $\sigma$ -distinguished arrows. To finish, it is enough to consider the cone  $\sigma = \langle p_0, p_{r-a}, p_r \rangle$  and observe that the only  $\sigma$ -distinguished arrows are  $x_0, \dots, x_{r-2}$ .

**Lemma 6.5.** *Let  $s, t \in Y$  be two points in the Danilov resolution, belonging to two distinct toric charts isomorphic to  $\mathbb{C}^3$ . Then the representations  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are not isomorphic.*

*Proof.* Let  $\sigma_s$  and  $\sigma_t$  be 3-dimensional cones in the fan of Danilov resolution corresponding to charts containing  $s$  and  $t$ . There are at most two common primitive generator of the cones  $\sigma_s$  and  $\sigma_t$  belonging to the set  $\{p_0, \dots, p_r\}$ . This implies that at least one  $y$ - or  $z$ -arrow is  $\sigma_s$ -distinguished and not  $\sigma_t$ -distinguished. Hence the representations  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are not isomorphic.

**Lemma 6.6.** *Let  $s, t \in Y$  be distinct points in the Danilov resolution, belonging to a single toric chart, isomorphic to  $\mathbb{C}^3$ , on the Danilov resolution. The representations  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are not isomorphic.*

*Proof.* Let  $\sigma = \langle p_i, p_j, p_k \rangle$  be the 3-dimensional cone in the fan of Danilov resolution, such that  $s, t \in U_\sigma$ , where  $U_\sigma$  stands for the toric chart given by  $\sigma$ . Let  $D_i^\sigma, D_j^\sigma, D_k^\sigma$  be restrictions of the divisors  $D_i, D_j, D_k$  to the chart  $U_\sigma$ . We claim that there exists  $i', j', k' \in \{0, \dots, r-1\}, i' \neq r-1$  such that the restriction of  $X_{i'}, Y_{j'}, Z_{k'}$  to the chart  $U_\sigma$  are equal to  $D_i^\sigma, D_j^\sigma, D_k^\sigma$ , respectively. This holds for  $a \in \{1, r-1\}$  and can be proven for  $a \notin \{1, r-1\}$  using recursion and the Lemmata (4.6), (4.5). In the orbit of the group  $\mathrm{GL}(Q, \mathbb{C})$  there exists exactly one representation, such that all  $\sigma$ -distinguished arrows are represented by the number 1 (by Lemma (6.4)). Therefore, in this unique element of the orbit, the arrows  $x_{i'}, y_{j'}, z_{k'}$  are represented by toric coordinates on  $U_\sigma$ . The points  $s, t$  have at least one toric coordinates different, therefore  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are not isomorphic.

**Corollary 6.7.** *For any two distinct points  $s, t \in Y$  in the Danilov resolution the representations  $\mathcal{F}_s$  and  $\mathcal{F}_t$  are not isomorphic.*

**Definition 6.8.** Let  $\mathcal{F}_i$  denote the representation of McKay quiver parameterized by the unique  $T$ -fixed point belonging to the toric chart  $U_{\sigma_i}$  (cf. Definition (3.4))

Since the divisors  $X_j - E_1$  are effective no  $x_j$ -arrow is  $\sigma_i$ -distinguished for any  $i$ . Moreover, by definition of permutation  $\tau$ , for any  $i$  there exists a unique  $j$ , such that  $y$ -arrow and  $z$ -arrow joining vertices  $j, j+a$  are not  $\sigma_i$ -distinguished. For  $i$  and  $j$  as above, if  $j' \neq j$  then among the  $y$ - and  $z$ -arrows joining  $j', j'+a$  exactly one is  $\sigma_i$ -distinguished. Hence the representations  $\mathcal{F}_i$  are particularly easy to deal with.

## 7. STABILITY OF QUIVER REPRESENTATIONS

In this section we recall some facts and definitions concerning  $\theta$ -stability of quiver representations (see [Kin94]). We prove that the representations in family  $\mathcal{F}$  (cf. Definition (6.2)) on the Danilov resolution are simultaneously  $\theta$ -(semi)stable if and only if the representation  $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$  are  $\theta$ -(semi)stable.

For any quiver  $Q$  set

$$\text{Wt}(Q) = \{ \theta : Q^0 \longrightarrow \mathbb{Q} \mid \sum_{v \in Q^0} \theta(v) = 0 \}.$$

Functions from  $\text{Wt}(Q)$  attaining only integral values can be identified with characters of  $\text{PGL}(\delta, \mathbb{C})$ . That is, for  $\theta \in \text{Wt}(Q)$ , we can define character  $\chi_\theta$  by setting

$$\chi_\theta(g) = \prod_{v \in Q^0} \det g(v)^{\theta(v)},$$

where  $g \in \text{PGL}(\delta, \mathbb{C})$ . Therefore, we will call  $\text{Wt}(Q)$  a weight space for  $Q$ .

**Definition 7.1.** For fixed  $a$  and  $r$  denote by  $\Theta$  the weight space  $\text{Wt}(Q)$  for the McKay quiver  $Q$  identified with  $r$ -tuples of rational numbers, that is

$$\Theta = \{ \theta \in \mathbb{Q}^r \mid \theta = (\theta_0, \dots, \theta_{r-1}), \sum \theta_i = 0 \}.$$

**Definition 7.2.** (A. King) For any representation  $V \in \text{Rep}(Q, \delta)$  and  $\theta \in \text{Wt}(Q)$  set

$$\theta(V) = \sum_{v \in Q^0} \theta(v) \delta(v).$$

Representation  $V$  is called  $\theta$ -semistable if for every proper, non-zero subrepresentation  $V' \subset V$ , with dimension vector  $\delta'$

$$\theta(V') \geq \theta(V) = 0.$$

Representation  $V$  is called  $\theta$ -stable if an analogous condition with strict inequality holds.

For a generic  $\theta$  the notions of  $\theta$ –stability and  $\theta$ –semistability coincide (see [Kin94]).

If arrows of a general quiver  $Q$  are represented by numbers (i.e. dimension vector  $\delta \equiv 1$ ), we can determine which subsets of vertices of  $Q$  form a subrepresentation.

**Lemma 7.3.** *Let  $V$  be a representation of a quiver  $Q$  with the constant dimension vector equal to 1. Then  $\delta'$  is a dimension vector of some subrepresentation  $V' \subset V$  if and only if for any arrow  $a \in Q^1$  the following condition holds:*

$$\text{if } \delta'(tl(a)) = 1, V(a) \neq 0 \text{ then } \delta'(hd(a)) = 1.$$

*Proof.* Follows directly from the definition of subrepresentation.

**Lemma 7.4.** *Let  $\theta$  be a stability parameter such that  $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$  are  $\theta$ –stable. Then for any  $T$ –fixed point  $s \in Y$  the representation  $\mathcal{F}_s$  is  $\theta$ –stable.*

*Proof.* Let  $s \in U_\sigma$  be a  $T$ –fixed point, where  $\sigma = \langle p_i, p_j, p_k \rangle$  is a 3–dimensional cone in the fan of the Danilov resolution and  $i < j < k$ . The proof is by induction on  $r$ . We will show that any sequence of vertices of McKay quiver, forming a subrepresentation  $V$  of  $\mathcal{F}_s$ , forms a subrepresentation (not necessarily isomorphic to  $V$ ) of some  $\mathcal{F}_j$ , where  $j \in \{0, \dots, r-1\}$ .

The theorem is trivial to check if  $a \in \{1, r-a\}$ . Assume that  $r > 1$  and the theorem is true for any  $r' < r$ . Let  $V$  be a subrepresentation of  $\mathcal{F}_s$ . By  $S(V) \subset \{0, \dots, r-1\}$  we mean a subset of the vertices of the McKay quiver, where the dimension vector of  $V$  is non-zero. Consider a sequence  $i, i + (r-a), \dots, i + s(r-a)$  of vertices in the set  $S(V)$ , such that the vertices  $i - (r-a), i + (s+1)(r-a)$  are not in  $S(V)$ . The set  $S(V)$  is a union of such sequences. There is no loss of generality in assuming that  $S(V)$  itself is a single sequence.

Suppose that  $z_{i-(r-a)}$ –arrow is  $\sigma$ –distinguished or the  $y_{i+(s+1)(r-a)}$ –arrow is  $\sigma$ –distinguished. We can assume that  $k < r+1$ , otherwise there is nothing to prove. The vertices  $i, \dots, i + s(r-a)$  form a subrepresentation of some of the representations  $\mathcal{F}_i, \dots, \mathcal{F}_{k-1}$ . This follows, say if  $z_{i-(r-a)}$ –arrow is  $\sigma$ –distinguished, from the fact that  $z_{i-(r-a)}$ –arrow is  $\sigma_l$ –distinguished and the  $z_{i+s(r-a)}$ –arrow is not  $\sigma_l$ –distinguished, for  $l = i, \dots, j-1$ .

Now we turn to the case when both  $z_{i+r-a}$ –arrow and  $y_{i+(s+1)(r-a)}$ –arrow are not distinguished. Assume that  $\sigma$  is an  $L$ –cone. Since  $y_i$ –arrow and  $z_{i+s(r-a)}$ –arrow are not  $\sigma$ –distinguished ( $V$  is a subrepresentation, see Lemma (7.3)), the sequence  $i, \dots, i + s(r-a)$  is concatenated out of some  $L$ –bricks. These  $L$ –bricks correspond to vertices of the McKay quiver for  $\frac{1}{r-a}(1, \bar{r}, -\bar{r})$ . Moreover, the vertices corresponding to these  $L$ –bricks form a subrepresentation of the representation  $(\mathcal{F}(r-a, \bar{r}))_s$  on the  $L$ –resolution. Now we can use the inductive assumption.

We state a general fact concerning families of quiver representations with constant dimension vector, equal to 1 on affine toric varieties.

**Lemma 7.5.** *Let  $U_\sigma$  be an affine toric variety containing a unique  $T$ -fixed point  $p$ . Let  $\mathcal{F}$  be a family of quiver representations on  $U_\sigma$ , with dimension function constant and equal to 1, given by a set of  $T$ -equivariant sections. If the representation  $\mathcal{F}_p$  is  $\theta$ -(semi)stable then any representation in  $\mathcal{F}$  is  $\theta$ -(semi)stable.*

*Proof.* The  $\theta$ -(semi)stability is an open condition and it is invariant under the  $T$ -action by the  $T$ -equivariance condition. Moreover, the  $T$ -fixed point lies in the closure of all orbits in  $U_\sigma$ .

**Corollary 7.6.** *If the representation  $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$  are  $\theta$ -(semi)stable then every representation in the family  $\mathcal{F}$  is  $\theta$ -(semi)stable.*

For the sake of completeness we gather some results concerning moduli of quiver representations.

**Theorem 7.7** (King). *Let  $Q$  be a quiver with dimension vector  $\delta$  and let  $\theta \in \text{Wt}(Q)$ . A point in  $\text{Rep}(Q, \delta)$  is  $\chi_\theta$ -(semi)stable under the action of  $\text{PGL}(\delta, \mathbb{C})$  if and only if the corresponding representation of  $Q$  is  $\theta$ -(semi)stable. Denote by  $M_\theta(Q)$  a GIT quotient of  $\text{Rep}(Q, \delta)$  by  $\text{PGL}(\delta, \mathbb{C})$  with respect to  $\chi_\theta$ -linearization of trivial bundle over  $\text{Rep}(Q, \delta)$ . That is,  $M_\theta(Q)$  is a projective variety constructed from the graded ring of semi-invariants, namely:*

$$M_\theta(Q) := \text{Rep}(Q, \delta) //_{\chi_\theta} \text{PGL}(\delta, \mathbb{C}) = \text{Proj} \bigoplus_{k=0}^{\infty} \mathbb{C}[\text{Rep}(Q, \delta)]^{\chi_\theta^k},$$

where  $\mathbb{C}[\text{Rep}(Q, \delta)]^\chi$  denotes ring of semi-invariants. Elements of the ring  $\mathbb{C}[\text{Rep}(Q, \delta)]^\chi$  are regular functions  $f$  on the representations space  $\text{Rep}(Q, \delta)$ , such that  $f(g \cdot v) = \chi(g)f(v)$ , for any  $g \in \text{PGL}(\delta, \mathbb{C})$  and any  $v \in \text{Rep}(Q, \delta)$ . The variety  $M_\theta(Q)$  is a coarse moduli space of  $S$ -equivalence classes of  $\theta$ -semistable representations of quiver  $Q$  (two  $\theta$ -semistable representations are  $S$ -equivalent if they have the same composition factors in Jordan-Hölder filtration (see [Kin94, §3])).

Moreover, if the dimension vector  $\delta$  is indivisible, the open subscheme of  $\theta$ -stable representations in  $M_\theta(Q)$  is a fine moduli space of  $\theta$ -stable representations.

*Proof.* See [Kin94, Propositions 3.1, 5.2, 5.3].

## 8. STABILITY OF THE REPRESENTATIONS $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$

We proved that every representation in the family  $\mathcal{F}$  family is  $\theta$ -(semi)stable if and only if the the representations  $\mathcal{F}_0, \dots, \mathcal{F}_{r-1}$  are simultaneously  $\theta$ -(semi)stable. We will show how to get such parameters  $\theta$  using permutation  $\tau$ .

**Definition 8.1.** Let  $\xi(r, a) = \tau(r, a)^{-1}$  denote an inverse of permutation  $\tau$  (see Definition (4.1)).

Since no  $x_i$ -arrow is  $\sigma_j$ -distinguished any two vertices of  $\mathcal{F}_j$  can be joined by a sequence of  $z$ - and  $y$ -arrows, by Lemma (6.4). Moreover, the arrows  $z_{\xi(r, a, j)}$  and  $y_{\xi(r, a, j) + (r - a)}$  are not  $\sigma_j$ -distinguished. Therefore, the quiver supporting representation  $\mathcal{F}_j$  consists of vertices  $0, 1, \dots, r - 1$  and every two vertices  $i, i + (r - a)$  are joined either by  $z$ -arrow or  $y$ -arrow (but not both) unless  $i = \xi(r, a, j)$ .

**Definition 8.2.** Assume that the vertices  $i, i + (r - a), \dots, \xi(r, a, j)$  of  $Q$  form a subrepresentation of  $\mathcal{F}_j$ . For all such  $i$  and  $j$  we denote this subrepresentation by  $V_{i, j}$ .

Assume that the vertices  $\xi(r, a, j) + (r - a), \dots, i + s(r - a)$  of  $Q$  form a subrepresentation of  $\mathcal{F}_j$ . For all such  $i$  and  $j$  we denote this subrepresentation by  $W_{i, j}$ .

The representation  $V_{i, j}$  is defined if  $z_{i - (r - a)}$ -arrow is  $\sigma_j$ -distinguished. The representation  $W_{i, j}$  is defined if the  $y_i$ -arrow is  $\sigma_j$ -distinguished.

**Lemma 8.3.** Let  $\theta \in \Theta$  be a fixed stability parameter. The representation  $\mathcal{F}_j$  is  $\theta$ -(semi)stable if and only if the numbers  $\theta(V_{i, j})$  and  $\theta(W_{i, j})$  are strictly greater (or equal) to zero for all  $V_{i, j}, W_{i, j}$  (whenever they are defined).

*Proof.* Let  $U$  be a subrepresentation of  $\mathcal{F}_j$  supported on vertices  $i, i + (r - a), \dots, i + s(r - a)$ . Then, by Lemma (7.3), the representations  $V_{i, j}$  and  $W_{i + s(r - a), j}$  are defined and  $\theta(U) = \theta(V_{i, j}) + \theta(W_{i + s(r - a), j})$ .

**Lemma 8.4.** Set  $\theta_i = n_i - n_{i + (r - a)}$  and let

$$\varphi(r, a, j) = \overline{\xi(r, a, j) + (r - a)}^r.$$

Then the representation  $\mathcal{F}_j$  are simultaneously  $\theta$ -stable if and only if

$$n_{\varphi(r, a, 0)} \leq n_{\varphi(r, a, 1)} \leq \dots \leq n_{\varphi(r, a, r - 1)}.$$

*Proof.* For fixed  $j$  fix some  $i \neq \xi(r, a, j)$ . Then either the  $z_{i - (r - a)}$ - or  $y_i$ -arrow is  $\sigma_j$ -distinguished. In the first case,  $V_{i, j}$  is well defined and  $\theta(V_{i, j}) = n_i - n_{\xi(r, a, j) + (r - a)} \geq 0$ . Otherwise  $W_{i - (r - a), j}$  is well defined and  $\theta(W_{i - (r - a), j}) = -n_i + n_{\xi(r, a, j) + (r - a)} \geq 0$ . By definition of permutation  $\tau$  and by definition of the divisors  $Y_i, Z_i$ , exactly  $r - 1 - j$  of  $z$ -arrows are  $\sigma_j$ -distinguished.

## 9. MAIN THEOREM

We defined a family of pairwise non-isomorphic representations of McKay quiver on the Danilov resolution  $Y$ , which are  $\theta$ -stable with respect to stability parameters  $\theta$  determined in Lemma (8.4). The universal property of the moduli space  $M_\theta(Q)$  will ensure that the Danilov resolution dominates one of its components.

**Definition 9.1** (Craw, Maclagan, Thomas). *Denote by  $Y_\theta$  (for generic  $\theta \in \Theta$ ) the unique irreducible component of the moduli  $M_\theta(Q)$ , containing representations of McKay quiver with all arrows represented by a non-zero number. Following [CMT07, Theorem 4.3], we call  $Y_\theta$  a coherent component of  $M_\theta(Q)$ .*

Note that representations of McKay quiver with all arrows represented by a non-zero number are  $\theta$ -stable under any stability condition  $\theta \in \Theta$ . The coherent component is reduced, irreducible, not-necessarily-normal toric variety of dimension 3, projective over  $X = \mathbb{C}^3/G$  (see [CMT07, Theorem 4.3]). Denote by  $\pi_\theta$  the corresponding projective, birational morphism:

$$\pi_\theta : Y_\theta \longrightarrow X.$$

By King [Kin94, Proposition 5.3], there exists a universal family of McKay quiver representations over the open set of  $\theta$ -stable representations in  $M_\theta(Q)$ . Denote by  $\pi$  natural toric morphism given by a sequence of toric weighted blowups

$$\pi : Y \longrightarrow X,$$

where  $Y$  denotes Danilov resolution.

Assume that the stability parameter  $\theta$  is chosen such that all representations of the McKay quiver in the family  $\mathcal{F}$  on  $Y$  are  $\theta$ -stable. Then there exists a unique morphisms  $\rho$

$$\rho : Y \longrightarrow M_\theta^s(Q) \subset M_\theta(Q),$$

where  $M_\theta^s(Q)$  denotes fine moduli of  $\theta$ -stable representations.

To show that  $\pi_\theta \circ \rho = \pi$  we need a particular result by T. Logvinenko proven in [Log08, Theorem 4.1]. The divisors  $R_i$  satisfy Logvinenko's *reductor condition*, hence, the family  $\bigoplus \mathcal{O}(-R_i)$  is a *gnat-family* (see [Log08, Definition 1.4]) which is in turn equivalent to the condition  $\pi_\theta \circ \rho = \pi$ .

**Theorem 9.2** (Main Theorem). *For any coprime natural numbers  $a, r$  and any rational numbers  $n_0, \dots, n_{r-1}$  such that*

$$n_{\varphi(r,a,0)} < \dots < n_{\varphi(r,a,r-1)},$$

*where  $\varphi(r, a, j) = \overline{\xi(r, a, j) - a}^r$  and  $\xi(r, a, \cdot)$  is an inverse of the permutation  $\tau(r, a, \cdot)$  (see Definition (4.1)), the Danilov resolution of the singularity of type  $\frac{1}{r}(1, a, r-a)$  is normalization of the irreducible component of the fine moduli space of  $\theta$ -stable representations of the McKay quiver containing representations corresponding to free orbits, for*

$$\theta = (\theta_0, \dots, \theta_{r-1}),$$

*where*

$$\theta_i = n_i - n_{i+(r-a)}.$$



*Proof.* By Lemma (8.4), every representation of McKay quiver in the family  $\mathcal{F}(r, a)$  on the Danilov resolution, constructed in Section 6, is  $\theta$ -stable for  $\theta$  taken as above. Therefore, there exists a unique morphisms  $\rho$

$$\rho : Y \longrightarrow M_\theta^s(Q) \subset M_\theta(Q),$$

where  $M_\theta^s(Q)$  denotes fine moduli of  $\theta$ -stable representations, and the following diagram commutes:

$$\begin{array}{ccc} Y & \xrightarrow{\rho} & Y_\theta \\ & \searrow \pi & \swarrow \pi_\theta \\ & X & \end{array}$$

Morphism  $\rho$  is proper since morphisms  $\pi$  and  $\pi_\theta$  are proper (see [Har77, Corollary II.4.8.(e)],  $\pi_\theta$  is projective, hence separated). By [Har77, Exercise II.4.4] the image of  $\rho$  in  $Y_\theta$  is closed and is of dimension 3 (see Corollary (6.7)).

By the work of A. Craw, D. Maclagan, R.R. Thomas [CMT07] the coherent component  $Y_\theta$  (i.e. containing representations with all arrows represented by a non-zero number) is a not-necessarily-normal toric variety of dimension 3, hence  $\rho$  is surjective onto  $Y_\theta$ . We are done if  $\rho$  is injective, that is if two representations of McKay quiver corresponding to two distinct closed points on the Danilov resolution are non-isomorphic. This is the content of Corollary (6.7).

**Example 9.3.** *The permutation  $\tau(5, 2)$  is a cycle of length 5, namely*

$$\tau(5, 2) = (0, 4, 1, 3, 2).$$

*Substituting consecutive integer numbers to the sequence*

$$n_0 < n_2 < n_1 < n_4 < n_3.$$

*gives  $\theta_0 = -4$ ,  $\theta_1 = -1$ ,  $\theta_2 = 1$  and  $\theta_3 = \theta_4 = 2$ .*

**Remark 9.4.** *Assume that result from [DH98],[Tha96], stating that a stability condition  $\theta$  belongs to a wall of a chamber if and only if there exists a strictly  $\theta$ -semistable point, extends to the case of action of a reductive group on a possibly-singular variety. By proof of Lemma (8.4), if some of the inequalities in the condition for  $n_i$  is not strict, then some representation  $\mathcal{F}_j$  is strictly  $\theta$ -semistable. Therefore, the condition of Theorem (9.2) defines interior of a chamber of stability conditions.*

Using computer algebra packages, the author have checked that for small values of  $a$  and  $r$  and random stability parameters as in Main Theorem (9.2) the moduli space of representation of McKay quiver is normal. This allows to conjecture that the coherent component is actually isomorphic to the Danilov resolution.

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